

# Hidden massive Dirac fermions in effective field theory for integral quantum Hall transitions

Yasuhiro Hatsugai\*

*Department of Applied Physics, University of Tokyo, 7-3-1 Hongo Bunkyo-ku, Tokyo 113, Japan*

Mahito Kohmoto

*Institute for Solid State Physics, University of Tokyo, 7-22-1 Roppongi Minato-ku, Tokyo 106, Japan*

Yong-Shi Wu

*Department of Physics, University of Utah, Salt Lake City, Utah 84112*

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Integral quantum Hall plateau transitions in a planar lattice system due to gap collapse can be described by an effective field theory with Dirac fermions. We discuss how to reproduce the correct integral values for the Hall conductance  $\sigma_{xy}$  before and after the plateau transition, which are dictated by the microscopic topological invariant. In addition to the massless Dirac fermions that appear at gap closing, the matching condition of  $\sigma_{xy}$  requires the introduction of massive Dirac fermions, as “spectators,” in the effective field theory. For noninteracting electrons on the lattice, we give a general prescription to determine these massive spectators, based on microscopic information on the vortices in the magnetic Brillouin zone which are closely related to edge states. Our description is demonstrated in a model with both nearest-neighbor and next-nearest-neighbor hoppings. [S0163-1829(96)03628-4]

## I. INTRODUCTION

Integral quantum Hall effect in a planar electron system is a beautiful manifestation of topological effects in physics. Whenever the Fermi level is in an energy gap, the quantized Hall conductance is known to be a topological invariant at the microscopic level.<sup>1-8</sup> For example, in a periodic system the Hall conductance for a filled band is the Chern number over the magnetic Brillouin zone which is topologically a torus,<sup>1,2</sup> usually called the TKNN invariant, named after Thouless, Kohmoto, Nightingale, and den Nijs. For a lattice system with edges, one of the authors (Y.H.) showed that the Hall conductance can be written as another topological invariant for the complex energy surface which is normally a higher-genus Riemann surface,<sup>7,8</sup> and there is a close relationship between the invariants and edge states,<sup>8</sup> which we are going to use in this paper. Thus the quantization of Hall conductance is expected to be stable against weak disorder and perturbations unless some topological change, e.g., gap closing, occurs.<sup>7-11</sup> When the gap is reopened, usually there is a discrete change in the Hall conductance. Such a Hall plateau transition due to gap collapse is an example of the so-called quantum phase transitions, which has recently attracted some theoretical attention.<sup>12-14</sup>

For definiteness, let us consider the model of noninteracting electrons on a square lattice, with both nearest-neighbor and next-nearest-neighbor hoppings, in a uniform magnetic field with rational flux,  $\phi = p/q$ , per plaquette. In general, the one-electron spectrum is split into bands. If the Fermi level is in a gap between two bands, the Hall conductance is quantized. At some special values for hopping parameters, the gap may close, leading to a Hall plateau transition.<sup>7,10</sup> One approach to studying such transitions is to use effective field theory. The basic idea is that in the neighborhood of gap closing points, energy-momentum dispersion is gener-

ally linear, corresponding to the appearance of massless Dirac fermion(s). Thus the large-distance behavior of the system near the transition can be described by an effective field theory with Dirac fermions, in which the gap closing and reopening are represented by sign change in fermion masses. (A similar situation is the anyon Mott transition near the fermion point, which has been studied in this way in Ref. 13.) Usually one is tempted to include in the effective theory only the massless fermions that appear when the gap closes; it is these fermions whose Dirac mass may change sign across the transition. Indeed, it has been generally shown that counting only these massless Dirac fermions at the transition point will give us the correct value for the discrete jump in Hall conductance across the transition.<sup>15</sup> Also only the massless Dirac fermions contribute to the critical exponents for the transition.<sup>13</sup> However, as we will see, this does not guarantee that the Hall conductance before and after the transition can be correctly reproduced. In other words, the topological nature of the Hall conductance dictates a matching condition for the large-distance effective field theory to be consistent with the microscopic lattice formulation.

To see the necessity of such consistency conditions, we note that the contribution of a planar massive Dirac fermion to Hall conductance is known to be  $\pm \frac{1}{2}$  in units of  $e^2/h$ , with the sign depending on the sign of Dirac mass. Therefore the change in Hall conductance is always an integer. However, if a total number of explicit Dirac fermions is odd, which is possible as we will see below in a numerical example, the Hall conductance is half-odd integer, in conflict with the integer quantization as required by the microscopic topological arguments. Thus a certain number of massive Dirac fermions, whose masses do not change sign across the plateau transition, must be present in the effective field theory. We will call such fermions “spectators,” since really they do not participate in the phase transition because they remain mas-

sive at the transition point. However, they are there in the effective theory, not only ensuring the Hall conductance to be integral, but also adding to the nontriviality of the problem, ensuring the integral value of the Hall conductance before or after plateau transition to be the *correct one* as dictated by the microscopic topological invariant. It is well known that if the massive Dirac fermions are not dynamically coupled to the Hall system, then they can just be removed by Lagrange multipliers; so they are irrelevant to the low energy dynamics of a system. The reader may wonder why they become relevant here. The point is that the massive Dirac fermions do have a nonvanishing contribution to the Hall current, in response to an *external electromagnetic probe*. The Lagrange multipliers cannot eliminate this response of theirs. Thus, while irrelevant to low-energy dynamics, massive spectators are of crucial importance in accounting for the Hall conductance, a response to an external electromagnetic probe. So the number of species of these massive spectators cannot be arbitrary; it needs to be determined from first principles. Previously the necessity of the existence of massive spectator fermions has been noticed in Refs. 13 and 16 in some simple and specific models for phase transitions to quantum Hall states. Here we will discuss a general prescription and a concrete procedure for determining all the massive spectators in more complicated models with arbitrary rational flux per plaquette.

The paper is organized as follows. First in the next section we briefly review the (2+1)-dimensional Dirac fermions, the associated zero mode and resulting contribution,  $\pm(\frac{1}{2})e^2/h$  per species, to Hall conductance, providing arguments for the necessity of having massive spectators in the theory. In Sec. III we propose a general prescription for determining the number of massive spectators, i.e., counting the number of “vortices” in the magnetic Brillouin zone for the highest filled band near the transition. A concrete procedure in which we find spectators by edge states is explained.<sup>8</sup> To exemplify, we use a tight-binding Hamiltonian with nearest-neighbor (NN) and next-nearest-neighbor (NNN) hoppings to illustrate our procedure. Section IV is devoted to exhibiting numerical results in this model, showing how our procedure works when gap closing occurs near certain critical values of the NNN hopping parameter. The existence of massive spectators is clearly demonstrated in the example. Finally in Sec. V we summarize our results and discuss the relationship of our Hall-conductance matching condition to precedents in the literature on matching topological property for theories at small and large distances.

## II. DIRAC FERMIONS IN (2+1) DIMENSIONS

First let us briefly review the Hall conductance for a system of planar Dirac fermions in the effective field theory, with “relativistic” energy-momentum relation, which becomes linear for the massless case. This quantity can be extracted from a simple calculation of the “vacuum-polarization” diagram for the photon propagator with a fermion loop. This one-loop diagram is superficially divergent, a certain regularization is needed. Then, the one-loop integral becomes finite even before removing the cutoff, with the result<sup>17,18</sup>

$$\sigma_{xy} = -\frac{1}{2} \sum_{\alpha} \text{sgn}(m_{\alpha}). \quad (2.1)$$

Here  $m_{\alpha}$  is the Dirac mass for species  $\alpha$ . It has been shown that if the fermions remain massive, there is no correction to the one-loop value of  $\sigma_{xy}$  from higher orders.<sup>19,20</sup> Other derivations without computing diagrams can be found in Refs. 21–23. Here we present an argument<sup>16,23</sup> which does not involve any diagram and regularization explicitly.

The hamiltonian for a planar “relativistic” fermion of species  $\alpha$  in a static, uniform, perpendicular magnetic field is given by

$$H_{\alpha} = \Pi_{\alpha}^1 \sigma^2 + \Pi_{\alpha}^2 \sigma^1 + m_{\alpha} \sigma^3, \quad (2.2)$$

where  $\Pi_{\alpha}^1$  and  $\Pi_{\alpha}^2$  are spatial components of covariant derivative, satisfying  $[\Pi_{\alpha}^1, \Pi_{\alpha}^2] = ie\hbar B$ ,  $B$  is the magnetic field strength, and  $\sigma^{(1,2,3)}$  are Pauli matrices. The spectrum for a single species at  $B=0$  is symmetric with respect to zero and is given by

$$\epsilon_{\alpha}^{\pm}(k) = \pm \sqrt{(\hbar k)^2 + m_{\alpha}^2} \quad (2.3)$$

which is gapless when  $m_{\alpha}=0$ , and is unbounded both from below and from above. When  $B \neq 0$ , relativistic Landau levels are given by

$$\epsilon_{\alpha,n}^{\pm} = \pm \sqrt{n\hbar|eB| + m_{\alpha}^2} \quad (n \geq 1), \quad (2.4)$$

$$\epsilon_{\alpha,0} = m_{\alpha} \text{sgn}(eB). \quad (2.5)$$

Note that there is a zero mode (2.5) labeled by  $n=0$ , which depends on the sign of fermion mass. For the vacuum state, whether this mode is filled or not depends on the sign of mass: filled if  $m_{\alpha} < 0$ , and not if  $m_{\alpha} > 0$ . When  $m_{\alpha}=0$ , there is a charge conjugation symmetry for this species. In the mean time, the vacuum is doubly degenerate, with the zero mode filled or not filled. In the continuum field theory, this ambiguity cannot be resolved. It has been argued<sup>23</sup> that a fermion-number fractionalization occurs if one assumes charge conjugation symmetry: the fermion number of the doubly degenerate vacuum is either  $+\frac{1}{2}$  or  $-\frac{1}{2}$  depending on whether the zero mode is filled or empty. Similarly for the Hall conductance, we may argue that since the contribution of the filled zero mode is always unity (in units of  $e^2/h$ ), charge conjugation symmetry requires that the vacuum with or without the zero mode filled should have fractional contribution  $\pm(\frac{1}{2})e^2/h$ , respectively. Now let us add an infinitesimal mass to the fermion to remove the vacuum degeneracy. Obviously for the vacuum state, the zero mode is filled if  $m_{\alpha} < 0$ , and empty if  $m_{\alpha} > 0$ . Thus we obtain the value of  $\sigma_{xy}$  to be  $\mp(\frac{1}{2})e^2/h$ , depending on the sign of mass. When there are several species of fermions, the Hall conductance is given by a sum of their individual contributions, as in Eq. (2.1).

Therefore, if the mass of a species changes sign across a plateau transition, it gives rise to a change in Hall conductance by unity. So to account for the change in Hall conductance, one only needs to look for massless fermions at the transition.

However, it is easy to see that if the number of massless fermions is odd at the transition, their total contribution to Hall conductance before or after the transition is a half-odd integer. If this gives the total Hall conductance, it is in conflict with integral quantization of Hall conductance implied by the TKNN topological invariant. One may wonder perhaps if this would never happen, similar to the no-go theorem of Nielsen and Ninomiya<sup>25</sup> in  $3+1$  dimensions, which asserts that in the continuum limit of a lattice theory the number of massless chiral fermions must be even. If a similar theorem could hold also in the  $(2+1)$ -dimensional case, perhaps the number of massless Dirac fermions must be even. Unfortunately, it is not the case: No such no-go theorem exists in  $2+1$  dimensions. Moreover, in Sec. IV we will show a numerical example in which gap closing occurs clearly at three isolated points in the magnetic Brillouin zone, corresponding to three (no more and no less) massless fermions.

Thus if the low-energy spectrum of the system consists of unpaired (massless) fermions, there must be massive fermions present in the “high-energy” sector of the theory. These massive fermions remain massive during the transition and therefore do not participate in the Hall plateau transition, in the sense that their contribution to Hall conductance does not change across the transition. Therefore they are spectators of the transition. But their existence restores the integral quantization of Hall conductance before or after the transition.<sup>13,16</sup> In the next section we will discuss the general principle, as well as a practical procedure, to determine the number of species of the “hidden” massive fermion spectators from microscopic information.

### III. VORTICITIES AND EDGE STATES IN THE NNN MODEL

For definiteness, let us consider a tight-binding Hamiltonian on a square lattice with both NN and NNN hoppings. In the Landau gauge, the Hamiltonian is given by

$$\begin{aligned} H = & -t \sum_{m,n} (c_{m+1,n}^\dagger c_{m,n} + c_{m,n+1}^\dagger e^{i2\pi\phi m} c_{m,n}) \\ & -t_d \sum_{m,n} c_{m+1,n+1}^\dagger e^{i2\pi\phi(m+1/2)} c_{m,n} \\ & -t'_d \sum_{m,n} c_{m,n+1}^\dagger e^{i2\pi\phi(m+1/2)} c_{m+1,n} + \text{H.c.}, \end{aligned} \quad (3.1)$$

where  $c_{m,n}$  is the annihilation operator for a lattice fermion at site  $(m,n)$ . We assume that the magnetic flux per plaquette  $\phi$  is rational, i.e.,  $\phi = p/q$  with mutually prime integers  $p$  and  $q$ .

The Hall conductance of a bulk system is described by the TKNN topological number.<sup>1</sup> It is written as the Chern number of a  $U(1)$  fiber bundle over the magnetic Brillouin zone as

$$\sigma_{xy}^{\text{bulk}} = -\frac{e^2}{h} \frac{1}{2\pi i} \int \int_{T_{\text{MBZ}}^2} dk_x dk_y [\nabla_k \times \mathbf{A}_u^j(\mathbf{k})]_z, \quad (3.2)$$

$$\mathbf{A}_u^j(\mathbf{k}) = \langle u^j(\mathbf{k}) | \nabla_k | u^j(\mathbf{k}) \rangle = \sum_{m=1}^q u_m^{j*}(\mathbf{k}) \nabla_k u_m^j(\mathbf{k}), \quad (3.3)$$

where  $u_m^j$  is a normalized Bloch function of the  $j$ th energy band, with  $m$  ( $1 \leq m \leq q$ ) labeling its components. (See references for precise definitions.<sup>2,8</sup>) An important observation here is that the magnetic Brillouin zone  $T_{\text{MBZ}}^2$  is topologically a torus rather than a rectangle. Since the torus does not have a boundary, application of Stokes's theorem to Eq. (3.2) would give  $\sigma_{xy} = 0$  if  $\mathbf{A}_u^j(\mathbf{k})$  is well defined on the entire torus  $T_{\text{MBZ}}^2$ . A possible nonzero value of  $\sigma_{xy}$  is a consequence of a nontrivial topology of  $\mathbf{A}_u^j(\mathbf{k})$ . In order to better understand the relevant topology, let us examine the effect of a phase transformation of the wave function

$$|u^j(\mathbf{k})\rangle' = e^{if(\mathbf{k})} |u^j(\mathbf{k})\rangle, \quad (3.4)$$

where  $f(\mathbf{k})$  is an arbitrary smooth function of  $\mathbf{k}$  over the Brillouin zone. The corresponding gauge transformation for  $\mathbf{A}_u^j(\mathbf{k})$  is

$$\mathbf{A}_u^j(\mathbf{k})' = \mathbf{A}_u^j(\mathbf{k}) + i \nabla_k f(\mathbf{k}), \quad (3.5)$$

which clearly leaves  $\sigma_{xy}$  invariant. The nontrivial topology arises when the phase of the wave function cannot be determined uniquely and smoothly in the entire Brillouin zone. The gauge transformation defined above implies that the overall phase of the wave function can be chosen arbitrarily. It can be determined, for example, by demanding the  $q$ th component of  $|u^j(\mathbf{k})\rangle$  be real. However, this is not enough to fix the phase over the Brillouin zone, when  $u_q^j(\mathbf{k})$  vanishes at some points. Let us denote these zeros by  $k_s^*$  with  $s = 1, \dots, N$ , and define small regions around the zeros by  $R_s^\epsilon = \{\mathbf{k} \in T_{\text{MBZ}}^2 | |\mathbf{k} - \mathbf{k}_s^*| < \epsilon, \Psi_q^j(\mathbf{k}_s^*) = 0\}$ ,  $\epsilon > 0$ . We may adopt a different phase convention in  $R_s$  so that another component, say,  $u_1^j(\mathbf{k})$ , is real. [We denote it by  $|v^j(\mathbf{k})\rangle$ .] Then the overall phase is uniquely determined on the entire Brillouin zone  $T_{\text{MBZ}}^2$ . At the boundaries,  $\partial R_s$ , we have a phase mismatch

$$|v^j(\mathbf{k})\rangle = e^{if(\mathbf{k})} |u^j(\mathbf{k})\rangle. \quad (3.6)$$

By using the above formulas for gauge transformation, Eqs. (3.4), (3.5), we have

$$\sigma_{xy}^j = - \sum_{s=1}^N n_s, \quad (3.7)$$

$$n_s = \frac{1}{2\pi} \oint_{\partial R_s} \nabla f(\mathbf{k}). \quad (3.8)$$

Here  $n_s$  must be integers since each of the state vectors must fit together exactly when we complete full revolutions around each  $R_s$ . This implies that the zeros of a certain component of the Bloch function define vortices in the Brillouin zone, whose integral vorticities contribute to the Hall conductance. While the phase of the wave function depends on phase convention (gauge choice), the total vorticity is a gauge invariant quantity. In this way, in principle, counting the total vorticity of the  $U(1)$  phase of the Bloch wave function gives the bulk Hall conductance. But this would need the knowledge of (explicit or numerical) wave functions in the whole Brillouin zone.

A more practical prescription to obtain the vortices of the bulk states is given in Refs. 7 and 8, which are summarized as follows. This method exploits a relationship between two closely related systems. We use periodic boundary conditions in  $y$  direction. But in  $x$  direction, we consider two possible boundary conditions separately: the periodic and fixed ones. Using the periodic boundary conditions in both directions and taking the infinite size limit, one can obtain a bulk system (without edges). By using the fixed boundary conditions in  $x$  direction, one can consider a cylindrical system with edges in that direction. In this case, by a Fourier transformation in  $y$  direction, we obtain a one-dimensional tight-binding equation with parameter  $k_y$ ,

$$\begin{aligned} & -(t + t_d e^{i\Lambda(m)} + t'_d e^{-i\Lambda(m)}) \Psi_{m+1}(k_y) - 2t \cos(-k_y) \\ & + 2\pi\phi m \Psi_m(k_y) - (t + t'_d e^{i\Lambda(m-1)} \\ & + t_d e^{-i\Lambda(m-1)}) \Psi_{m-1}(k_y) = E(k_y) \Psi_m(k_y), \end{aligned} \quad (3.9)$$

where

$$\Lambda(m) = -k_y + 2\pi\phi m + \pi\phi, \Lambda(m+q) = \Lambda(m). \quad (3.10)$$

This difference equation can be written as

$$\begin{pmatrix} \Psi_{m+1} \\ \Psi_m \end{pmatrix} = M_m(E, k_y) \begin{pmatrix} \Psi_m \\ \Psi_{m-1} \end{pmatrix}, \quad (3.11)$$

where  $M_m$  is the following  $2 \times 2$  transfer matrix:

$$M_m(E, k_y) = \begin{pmatrix} \frac{-E - 2t \cos(-k_y + 2\pi\phi m)}{t + t_d e^{i\Lambda(m)} + t'_d e^{-i\Lambda(m)}} & 1 \\ \frac{t + t'_d e^{i\Lambda(m-1)} + t_d e^{-i\Lambda(m-1)}}{t + t_d e^{i\Lambda(m)} + t'_d e^{-i\Lambda(m)}} & 0 \end{pmatrix}. \quad (3.12)$$

Then the spectrum is completely governed by a product of  $q$  transfer matrices,  $M(E, k_y)$ , given by

$$M(E, k_y) = \prod_{m=1}^q M_m(E, k_y), \quad |\det M(E, k_y)| = 1. \quad (3.13)$$

In the following, for simplicity, we assume  $t = 1$ ,  $t'_d = t_d$ , and  $L_x$  to be a multiple of  $q$ . The boundary conditions are  $\Psi_0 = \Psi_{L_x} = 0$  and we set  $\Psi_1 = 1$  to fix the normalization.

The one-dimensional system (3.9) has a period  $q$ . Thus as  $L_x \rightarrow \infty$ , part of the spectrum converges to  $q$  energy bands which are determined by the condition

$$|\text{Tr} M(E, k_y)| \leq 2. \quad (3.14)$$

This corresponds to the on-shell condition in usual scattering theory. Further, for the cylindrical system, there are additional eigenenergies,  $\mu_j$  ( $j = 1, \dots, q-1$ ), satisfying

$$M_{12}(\mu_j, k_y) = 0, \quad (3.15)$$

so that  $\Psi_{L_x} = 0$ . For each  $\mu_j$ , the condition (3.14) is not satisfied. On the other hand, the corresponding state is localized near either  $x \approx 0$  or  $x \approx L_x$ , depending on the value of  $|M_{22}(\mu_j, k_y)|$ :

$$|M_{22}(\mu_j, k_y)| < 1, \quad \text{localized at the left edge,}$$

$$|M_{22}(\mu_j, k_y)| > 1, \quad \text{localized at the right edge.} \quad (3.16)$$

So this is an edge state, whose energy lies in the  $j$ th gap. Note that for such a state, the above equation implies

$$\Psi_q(\mu_j(k_y), k_y) = 0. \quad (3.17)$$

On the other hand, if  $|M_{22}(\mu_j(k_y), k_y)| = 1$ , the condition (3.14) is satisfied, implying that the edge state is in touch with an energy band edge. Then the above equation (3.17) tells us that the corresponding  $k_y$ , together with  $k_x = 0$  or  $2\pi/q$  depending on which (top or bottom) edge it is of the energy band, gives a zero of the Bloch function  $\Psi_q$  in the Brillouin zone. In this way, each time the edge state becomes degenerate with a bulk state at band edge, we find a zero of the  $q$ th component of the Bloch wave function for that band, which gives a desired  $U(1)$  vortex in Eq. (3.2).

How to determine the vorticity associated with each vortex found in this way? A useful and practical rule is proved in Ref. 8: When the degeneracy occurs at the top of a band and the edge state jumps from the right edge to the left (or the opposite way from the left edge to the right) with increasing  $k_y$ , the vorticity is  $+1$  (or  $-1$ ). Similarly, when the degeneracy occurs at the bottom of a band, the vorticity is just opposite to the previous case. In Fig. 1, a typical example of the degeneracy between an energy band and edge state is shown. The energy of an edge state that is localized at the right edge is drawn by a solid line and the one at the left edge by a dotted line. Since there is one and only one edge state in the  $j$ th gap, we may trace its energy  $\mu_j(k_y)$  as  $k_y$  varies from 0 to  $2\pi$ , and define a winding number (around the  $j$ th gap) associated with the edge state. [Geometrically indeed this number can be realized<sup>8</sup> as the winding number of the loop traced by  $\mu_j(k_y)$  around the  $j$ th hole on the complex energy Riemann surface for Eq. (3.9).] This winding number is exactly the contribution of the  $j$ th edge state to the Hall conductance in an edge state picture (see Ref. 7 and 8). In this way, a direct relationship between the bulk Hall conductance and the behavior of the edge state was established in Ref. 8. We will exploit this relation to find the vorticities of the bulk Bloch states. As will be discussed in the next section, our rule is simply to assign a Dirac fermion to each vortex associated with the edge state in the gap relevant to the plateau transition.

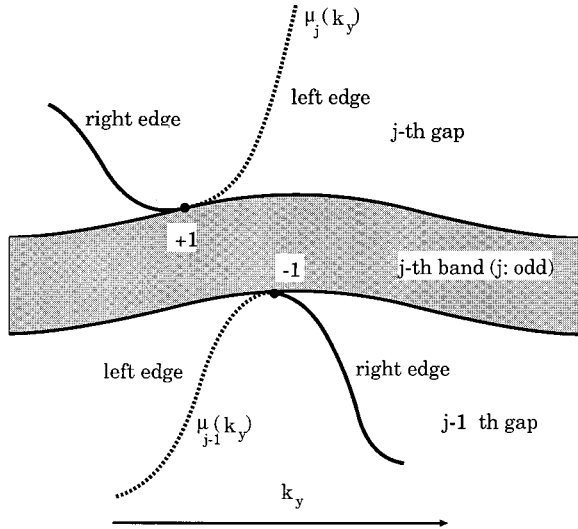


FIG. 1. Typical example for the degeneracy between the energy band and the edge states. It gives the vortex.

#### IV. DIRAC FERMIONS AND SPECTATORS ON THE LATTICE

In the model with only the NN hopping, it is known that there are zero modes when  $q$  is even.<sup>9–11</sup> There are  $q$  zero modes and the dispersion is linear in momentum near these zero modes. As far as the low energy physics is concerned, when the Fermi energy is near zero, we may treat the system as that of Dirac fermions. The energy dispersion for the  $\phi = \frac{1}{4}$  case is shown in Fig. 2(a). There are four energy bands and the two of them near  $E=0$  are degenerate where the linear dispersion is clearly observed.

Dirac fermions also appear in the NNN model.<sup>10,15</sup> For example, in Fig. 2(b), the spectrum for  $\phi = \frac{1}{3}$  and  $t_d = t_{\text{Dirac}}$ ,  $t_{\text{Dirac}} \approx 0.2679t$  is shown. In this case, there are three energy bands and the two higher energy bands are degenerate at three momenta, corresponding to three massless Dirac fermions.

In general, there are  $q$  gap closing points in the NNN model with  $t_d = t_{\text{Dirac}}$ . Deviation of  $t_d$  from  $t_{\text{Dirac}}$  brings a mass term in the energy dispersion. The formula (2.1) tells that one Dirac fermion carries the Hall conductance  $\pm \frac{1}{2}$  ( $e^2/h$ ) depending on the sign of its mass. For an even number of Dirac fermions [as shown in Fig. 2(a)], Eq. (2.1) always gives an integer  $\sigma_{xy}$ . On the other hand, if  $q$  is odd, according to Eq. (2.1), these  $q$  Dirac fermions associated with the gap closing points will give a half-integer value for any kind of combination of signs in the mass terms. It would break the integral quantization for the Hall conductance of the noninteracting fermions if no massive Dirac fermions are introduced. Indeed, as will be shown below, for example, for the case of Fig. 2(b), there is in addition a massive Dirac fermion, corresponding to a vortex not passing through any of the gap closing points. Though it is a spectator for this Hall transition it helps restore the integral Hall conductance.

In Fig. 3, the energy bands, edge states, and the corresponding vortices of the NN model are shown where the white circles denote +1 vorticities and the black ones denote

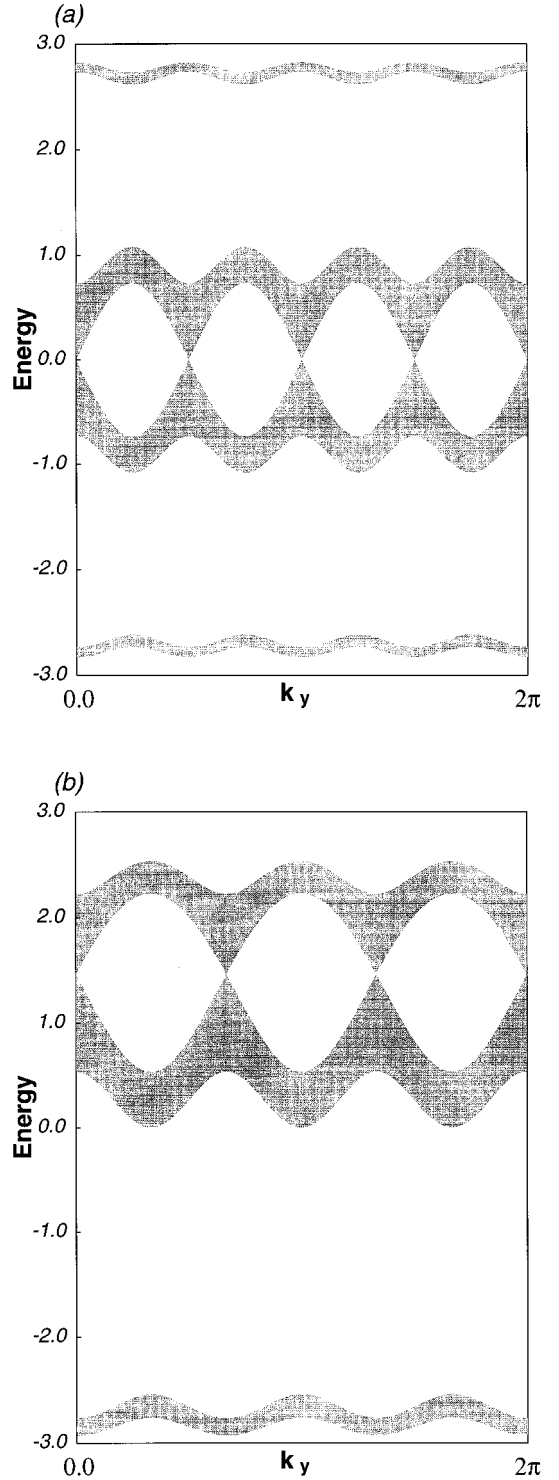


FIG. 2. Energy spectrum of the two-dimensional tight-binding model: (a)  $\phi = \frac{1}{4}$  and  $t_d = 0.0$ , (b)  $\phi = \frac{1}{3}$  and  $t_d = t_{\text{Dirac}}$ ,  $t_{\text{Dirac}} \approx 0.2679$ .

−1 vorticities. In Fig. 4, the same things are shown for the NNN model for several  $t_d$ 's. One can see how the second energy gap closes and then opens again. The energy dispersions near the gap closing are shown in Fig. 4(d) for  $t_d = 0.267945t$  and Fig. 4(e) for  $t_d = 0.267955t$ , where there are three almost linear dispersions in the second energy gap. The exact linear dispersion is realized at  $t_d = t_{\text{Dirac}} \approx 0.26795$ . Thus the Dirac fermions are massive in the cases

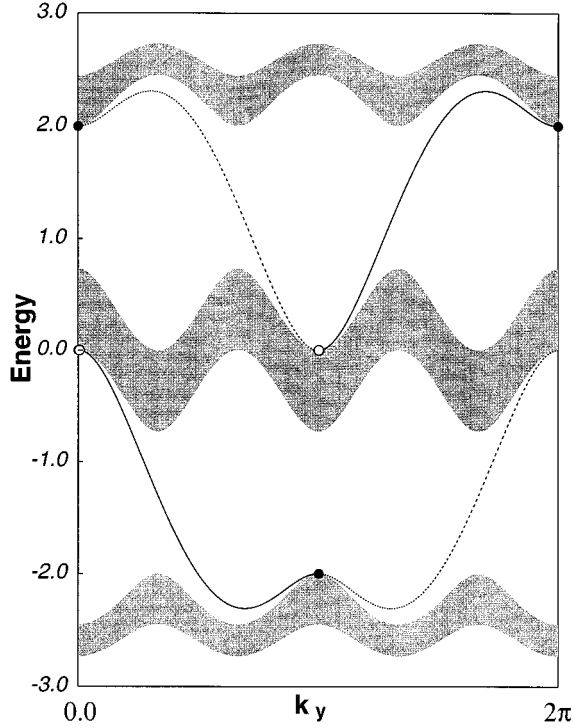


FIG. 3. Energy spectrum of the two-dimensional tight-binding model with  $t_d=0$  and  $\phi=\frac{1}{3}$  (NN model). The energy of the edge states and the corresponding vortices on the energy bands are also shown. The solid line is the edge state localized at the right edge and the dotted line is one localized at the left edge. The vorticities are denoted by the white or black circles. The black one is vorticity  $+1$  and the white one is  $-1$ .

of Figs. 4(d) and 4(e). Further, by comparing Figs. 4(c) and 4(d),<sup>10</sup> one can see that the sign of the mass is reversed after the gap reopening. Near the gap closing (appearance of the massless Dirac fermions) at the three different momentum points, there are three vortices as shown in Fig. 4. The vortices move between the lower band and the higher band during the process. This explains the change in the Hall conductance.

What is crucial to our point is that there is an additional vortex near the  $k_y=\pi$  point, where there is a large energy gap. Though the shape of the energy dispersion is quite different from that of a Dirac fermion, it does not prevent us from including the effect of this vortex as a Dirac fermion with a large mass term in the effective field theory, since its only role in the theory for the Hall transition is to reproduce the correct value of the Hall conductance.<sup>24</sup> This is the hidden spectator which restores the integral quantization of Hall conductance. In short, it is the existence of the vortices rather than their energy dispersion that defines all relevant Dirac fermions appearing in the effective theory, including massive spectators.

Note that when one changes the parameters of the model, the Dirac fermions associated with vortices in the magnetic Brillouin zone are always created in pairs. For example, in Figs. 4(a) and 4(b), between  $t_d=0.24$  and  $0.250\,03$ , a pair of

Dirac fermions are created near  $k_y=\pi$ . Two fermions move toward the gap closing points at  $k_y=2\pi/3$  and  $4\pi/3$  and there they go across the energy gap when the gap closes [Figs. 4(a)–4(g)]. In summary, for the effective theory of this quantum Hall transition, we should have three massless Dirac fermions and one massive spectator with positive mass.

## V. SUMMARY AND DISCUSSION

We have shown that in general one needs to add massive fermions (spectators) to the effective field theory in order to get correct integral Hall conductance. The number of spectators is not arbitrary, since the Hall conductance is dictated by the microscopic topological invariants. Therefore one needs to consider the “high-energy” sector or the microscopic details of the system, in order that the Hall conductance in the low-energy effective field theory at large distances should be the same as in the high-energy microscopic model at small distances. This matching condition, required by topological considerations, establishes a connection between theories of the same system at large and small distances. In the models with noninteracting electrons on a planar lattice, we have given the principles and a practical procedure for determining the number of massive species of fermion spectators from microscopic information. Numerical examples are shown for the validity of our prescription and for the implementation of our practical procedure. Application of the present arguments to the fractional quantum Hall effect is an interesting future problem.

To conclude this paper, we would like to point out the relationship of our present problem to its precedents in the literature on topological investigations in quantum field theory. From the point of view of the latter, the Hall effect in a planar system (or in  $2+1$  dimensions) can be expressed by a Chern-Simons effective action, obtained by integrating out the electrons in the system, for the external electromagnetic field:

$$S_{\text{eff}}(A) = \frac{\sigma_{xy}}{2} \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda}, \quad (5.1)$$

whose variation with respect to  $A_\mu$  gives us the (Hall) current in the system induced by the external field. (Here  $\mu, \nu, \lambda=0,1,2$ .) There is a well-known formal relationship<sup>21,26</sup> between this action and the famous axial anomaly equation in  $3+1$  dimensions:

$$\partial_\mu J_5^\mu = \frac{1}{16\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}. \quad (5.2)$$

Here nonconservation of the axial current,  $J_5^\mu$ , is expressed in terms of the gauge field that couples to the vector current. One observes that the right side of Eq. (5.1) can be viewed, up to numerical factors, as the surface term of an integral of

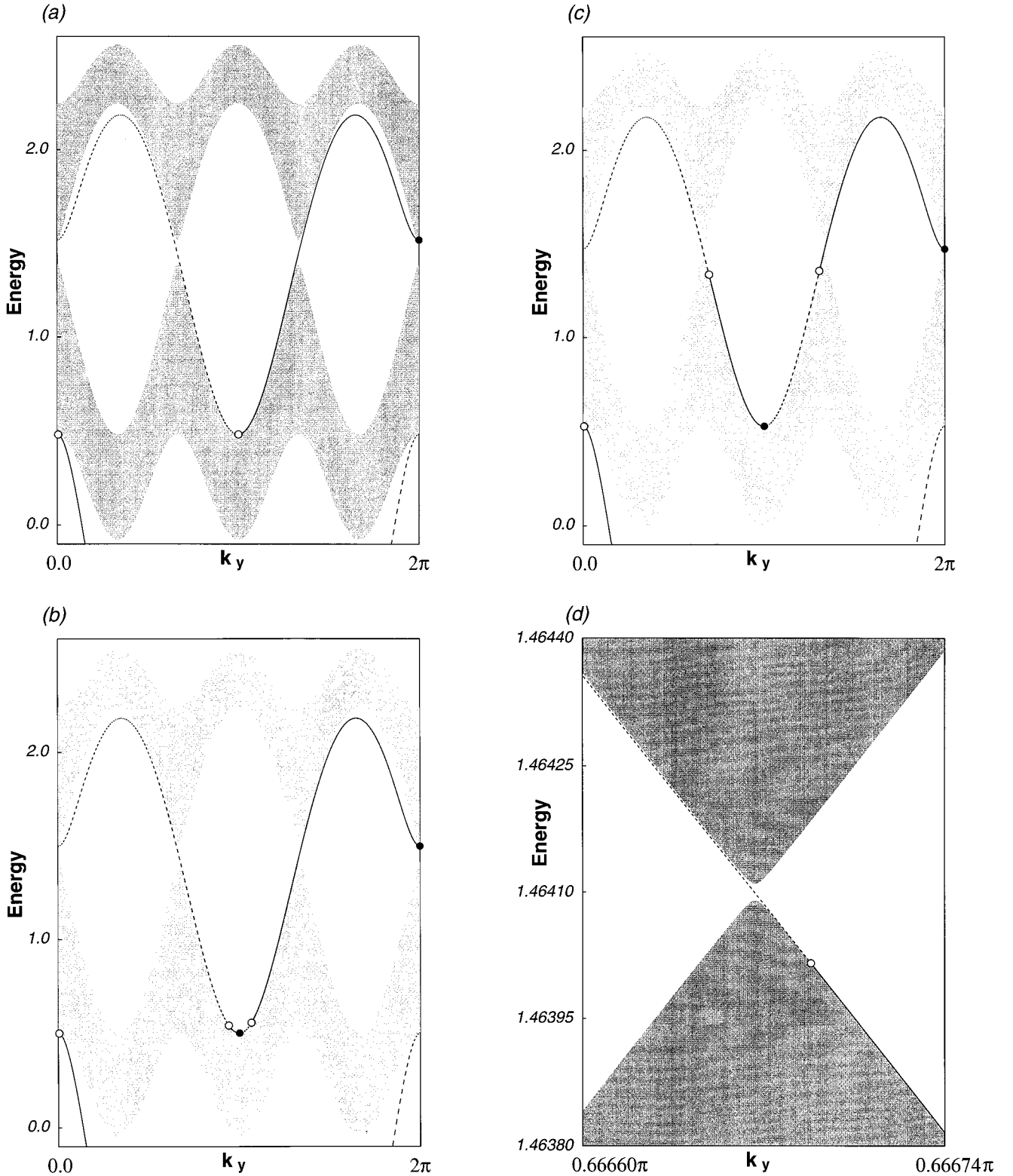


FIG. 4. Energy spectrum of the two-dimensional tight-binding model with next-nearest-neighbor hopping  $t_d$  (NNN model). The energy of the edge states and the corresponding vortices on the energy bands are also shown for  $\phi = \frac{1}{3}$ : (a)  $t_d = 0.24$ , (b)  $t_d = 0.250\,03$ , (c)  $t_d = 0.263$ , (d)  $t_d = 0.267\,945$ , (e)  $t_d = 0.267\,955$ , (f)  $t_d = 0.273$ , and (g)  $t_d = 0.4$ .

the right side of Eq. (5.2). It is not surprising to find similarities between our present problem and some known examples related to the axial anomaly in 3+1 dimensions.

The first example we like to mention is t'Hooft's

anomaly-matching condition.<sup>27</sup> In 3+1 dimensions, chiral fermions give rise to axial anomaly, which is known to be of topological origin. If this anomaly exists at the fundamental constituent level, t'Hooft has argued that it must survive at

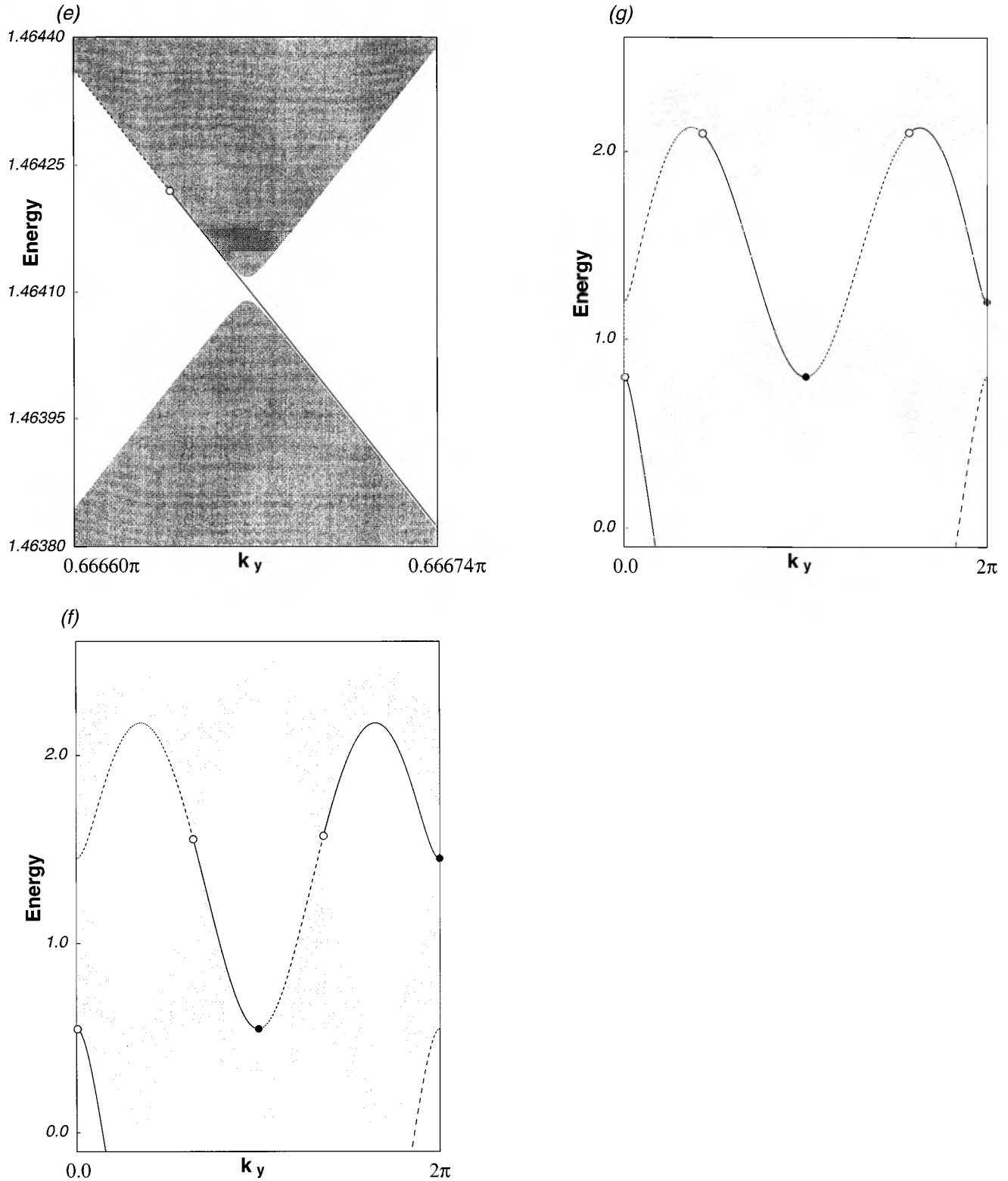


FIG. 4. (Continued).

large distances even in theories in which the massless fermions are confined (like quarks) and do not show up at large distances. This requirement, i.e., that the axial anomaly in the effective theory at large distances should match that in the fundamental theory at small distances, imposes constraints on the spectrum of the effective theory at large distances. In certain cases, massive spectators may be needed to

satisfy this condition. In our problem, what plays the role of axial anomaly is the Hall conductance in 2+1 dimensions. The lattice theory we start with is the microscopic theory and the effective field theory is the one at large distances.

In a certain sense, the Nielsen-Ninomiya no-go theorem for unpaired massless chiral fermions in the continuum limit of a lattice theory constitutes another counterpart of our



problem in 3+1 dimensions. The basic argument for this no-go theorem is again the necessity of matching, in the continuum limit, the vanishing axial anomaly in the lattice theory.

Therefore our present problem can be viewed as a (2+1) dimensional version of the above two problems for matching topological properties for theories of the same system at small and large distances.

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\*Electronic address: hatsugai@coral.t.u-tokyo.ac.jp.

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